Truth in Fiction:

Origins and Consequences of Leibniz's Doctrine of Infinitesimal Magnitudes.

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Abstract: This paper investigates the background to Leibniz's doctrine of the fictionality of infinitesimal magnitudes and the consequences the doctrine has for his account of the foundations of the calculus. It first traces the connection between Leibniz's doctrine of "incomparably small" magnitudes and Hobbes's doctrine of *conatus*, particularly as it is applied to the study of geometric figures. The concluding sections consider the application of this doctrine to disputes about the reality of infinitesimal magnitudes.

The status of infinitesimals in Leibniz's philosophy of mathematics is an issue whose resolution is not without difficulty. In many contexts Leibniz's account of his *calculus differentialis* is phrased in terms that are most readily interpreted as implying the real existence of infinitely small magnitudes. In other places, he claims that there are, in actual fact, no infinitely small magnitudes and the device of infinitesimals is simply a convenient fiction, useful for stating and deriving results, but without any serious ontological import. One can therefore sensibly ask whether Leibniz truly believed in the reality of infinitesimal magnitudes, but thought that the central results of his *calculus differentialis* might be formulated and derived by means that did not presuppose the reality of the infinitesimal. Pursuing this sort of interpretive strategy would obviously require that Leibniz's frequent claims about the fictionality of the infinitesimal be taken a something less than face value. That is not a decisive problem for an interpretation of Leibniz as a realist about infinitesimals, but it does suggest that one ought at least to consider the prospects for seeing Leibniz as committed to the view that the infinitesimal has the status of a "well founded fiction." I take Leibniz's claims about the fictionality of the infinitesimal to be his considered view on the subject, although I am not convinced that he held consistently to a "fictionalist" position from his earliest writings on the calculus.

My purpose here is to trace what I take to be the origins of Leibniz's notion of the fictional infinitesimal, which I believe can be found in Hobbes's doctrine of *conatus*, and particularly the application that Hobbes made of this concept in the solution of geometric problems of tangency, quadrature and arclength determination – precisely the sorts of problems that the Leibnizian calculus was designed to solve. Having shown the role that the *conatus* concept plays in Hobbes's approach to mathematics, I will argue that some salient features of it appear in Leibniz's formulation of the calculus. In particular, the notion that *conatus* is a finite, but negligibly small, quantity is significant. Ultimately, I think that Hobbes's notion of *conatus*, or at least a near descendent of it, appears in Leibniz's claim that infinitesimal magnitudes are "well founded fictions" that can, in principle, be replaced by the consideration of finite quantities. In this context, Leibniz's notion of "incomparably small" quantities plays a central role, and I think it can be shown that the Leibnizian theory of the incomparably small (yet finite) magnitude has its roots in the Hobbesian of *conats*. I will briefly outline the role *conatus* in the Hobbesian approach to geometry; with this material in hand, I will investigate some of Leibniz's pronouncements on the foundations of his calculus with the aim of showing that these owe a significant debt to Hobbes's proposals.

1. Hobbes, *Conatus*, and the Mathematics of Motion

Hobbes first introduced the concept of *conatus* in his 1655 treatise *De Corpore* -- a work presented as the first part of the elements of philosophy and containing Hobbes's doctrines on the nature of body as well as his exposition of a thoroughly materialistic philosophy of mathematics. As Hobbes defines it, *conatus* is essentially a point motion, or motion through an indefinitely small space: "*conatus*" he declares, "is motion through a space and a time less than any given, that is, less than any determined whether by exposition or assigned by number, that is, through a point." (Hobbes [1839-45] 1966a, 1: 177) Hobbes employs his idiosyncratic conception of points here, in which a point is an extended body, but one sufficiently small that its magnitude is not considered in a demonstration. In explicating the definition of conatus he therefore remarks that "it should be recalled that by a point is not understood that which has no quantity, or which can by no means be divided (for nothing of this sort is in the nature of things), but that whose quantity is not considered, that is, neither its quantity nor any of its parts are computed in demonstration, so that a point is not taken for indivisible, but for undivided. And as also an instant is to be taken as an undivided time, not an indivisible time."

(Hobbes [1839-45] 1966a 1: 177-8) The result is that conatus is a kind of "tendency toward motion" or a striving to move in a particular direction.

This definition allows for a further concept of *impetus*, or the instantaneous velocity of a moving point; the velocity of the point at an instant can be understood as the ratio of the distance moved to the time elapsed in a conatus. In Hobbes's terms "impetus is this velocity [of a moving thing] but considered in any point of time in which the transit is made. And so impetus is nothing other than the quantity or velocity of this conatus." (Hobbes [1839-45] 1966a 1:178) The concepts of *conatus* and *impetus* are basic to Hobbes's analysis of motion, and it is not great exaggeration to say that his whole program for natural philosophy, which he deemed the true science of motion, is drawn from his account of *conatus* and *impetus*.

The concepts of *impetus* and *conatus* can be applied to the case of geometric magnitudes as well as to moving bodies. Because Hobbes held that geometric magnitudes are generated by the motion of points, lines, or surfaces, he also held that one could inquire into the velocities with which these magnitudes are generated, and this inquiry can be extended to the ratios between magnitudes and their generating motions. For example, take a curve to be traced by the motion of a point, and at any given stage in the generation of the curve, this generating point will have a (directed) instantaneous velocity. This, in turn, can be regarded as the ratio between the indefinitely small distance covered in an indefinitely small time; this ratio will be a finite magnitude which can be expressed as the inclination of the tangent to the curve at the point. Consider, for instance the curve αβ as in figure 1. The *conatus* of its generating point at any instant will be the "point motion" with which an indefinitely small part of the curve is generated;

the impetus at any stage in the curve's production will be expressed as the ratio of the distance covered to the time elapsed in the conatus. Represent the time by the x-axis and the distance moved by the y-axis. Then (assuming time to flow uniformly), the instantaneous impetus will be the ratio between the instantaneous increment along to the y-axis to the increment along the x-axis. The tangent to the curve at the point *p* is the right line that continues or extends the *conatus* a *p*; or, equivalently, the tangent is the dilation or expansion of the point motion into a right line.

It is important to observe here that the tangent is constructed as a finite ratio between two quantities that, in themselves, are small enough to be disregarded. That is to say, the ratio between two "inconsiderable" quantities may itself be a considerable quantity. Hobbes emphasizes this feature of his system when he stresses that points may be larger or smaller than one another, although in themselves they are quantities too small to be considered in a geometric demonstration. Thus, in discussing the comparisons that may be made between one conatus and another, Hobbes declares: "as a point may be compared with a point, so a conatus can be compared with a conatus, and one may be found to be greater or less than another. For if the vertical points of two angles are compared to one another, they will be equal or unequal in the ratio of the angles themselves to one another; or if a right line cuts many circumferences of concentric circles, the points of intersection will be unequal in the same ratio which the perimeters have to one another." ([1839-45] 1966a, 1: 178)

Hobbes's concepts of *conatus* and *impetus* can also be applied to the general problem of quadrature by analyzing the area of a plane figure as the product of a moving line and time. Hobbes himself was eager to solve problems of quadrature (most notably the quadrature of the circle), and it is here that his concept of conatus is put most fully to work. Indeed, it is no exaggeration to say that the third part of *De Corpore* (which bears the title "On the Ratios of Motions and Magnitudes") is Hobbes's attempt to furnish a general method for finding quadratures. In the very simplest case, the whole impetus imparted to a body throughout a uniform motion is representable as a rectangle, one side of which is the line representing the instantaneous impetus while the other represents the time during which the body is moved. More complex cases can then be developed by considering non-uniform motions produced by variable impetus. In chapters 16 and 17 of *De Corpore* Hobbes approached a variety of different quadrature and tangency problems, and in so doing he presented a number of important results that belong to the "pre-history" of the calculus. Of special interest in this context is Hobbes's appropriation of important results from Bonaventura Cavalieri's *Exercationes Geometricae Sex*, which he set forth in chapter 17 of *De Corpore* as an investigation into the area of curvilinear figures.

[Figure 2]

The subject of chapter 17 is "deficient figures," and it presents something very much like an early analysis of integration. In Hobbes's parlance the deficient figure *ABDGA* in figure 2 is produced by the motion of the right line *BD* through *BA*, while *BD* diminishes to a point at *A*. The "complete figure" corresponding to the deficient figure is the rectangle *ABDC*, produced by the motion of *BC* through *AB* without diminishing. The complement of the deficient figure is *DGAC*, the figure that, when added to the deficient figure, makes the complete figure. Hobbes proposes to determine the ratio of the area of the deficient figure to its complement, given a specified rate of decrease of the quantity *BD*. He concludes that the ratio of the deficient figure to its comp0lement is the same as the ratio between corresponding lines in the deficient figure and their counterparts in the complement. As he states the theorem in article 2 of chapter 17:

> A deficient figure, which is made by a quantity continually decreasing until it vanishes, according to ratios everywhere proportional and commensurable, is to its complement as the ratio of the whole altitude to an altitude diminished at any time is to the ratio of the whole quantity which describes the figure, to the same quantity diminished in the same time.

Thus, if the rate of diminution of *BD* is uniform the line AD will be a right line (the diagonal of the rectangle), and the deficient figure will be to its complement in the ratio of one to one. In more complex cases, as when *BD* decreases as the square of the diminished altitude, the area of the deficient figure will be twice that of its complement. And, in general, if the line *BD* decreases as the power *n*, the ratio of the deficient figure to its complement will be *n*:1.

In the fourth of his six *Exercationes Geometricae* Cavalieri pursued a result that historians of mathematics generally characterize as the attempt to prove the geometric equivalent of the theorem that the integral from zero to *a* of $x^n dx$ is equal to $a^{(n+1)}/(n+1)$. Except for differences in diagrams and terminology, Cavalieri's fourth *Exercitatio* delivers the same results as Hobbes's account of deficient figures. The central theorem, which is the analogue of the result we just saw stated by Hobbes, reads:

[Figure 3]

> In any parallelogram such as *BD* [as in Figure 2] with the base *CD* as *regula*, if any parallel to *CD* such as *EF* is taken, and if the diameter *AC* is drawn, which cuts the line *EF* in *G*, then as *DA* is to *AF*, so *CD* or *EF* will be to *FG*. And let *AC* be called the first diagonal. and again as DA^2 is to AF^2, let *EF* be to *FH*, and let this be understood in all the parallels to *CD*, so that all of these homologous lines *HF* terminate in the curve *AHC*. Similarly, as DA^3 is to AF^3, let also *EF* be to *FI*, and likewise in the remaining parallels, to describe the curve *CIA*. And as AD^4 is to AF^4, let *EF* be to *FL*, and likewise in the remaining parallels to describe the curve *CLA*. Which procedure can be supposed continued in other cases. Then *CHA* is called the second diagonal, *CIA* the third diagonal, *CLA* the fourth diagonal, and so forth. Similarly the triangle *AGCD* is called the first diagonal space of the parallelogram, the trilinear figure *AHCD* is the second diagonal space of the parallelogram, *AICD* the third, *ALCD* the fourth, and so on. I say therefore that the parallelogram *BD* is twice the first space, triple the second space, quadruple the third space, quintuple the fourth space, and so forth. (Cavalieri 1647, 279).

Hobbes and Cavalieri employed different proof procedures in attempting to establish this result. Although I lack the time to go into these in detail, it is worth observing that Hobbes's procedure (at least in some of its guises) employs the idea of a *conatus* or the "aggregate of the velocities" whereby lines in a figure are generated. There is enough similarity between Hobbes and Cavalieri here to warrant the conclusion that Hobbes borrowed quite heavily from the Italian mathematician. Nevertheless, Hobbes did re-cast some of Cavalieri's language in a way that emphasizes the consideration of point motion or *conatus*, and Hobbes evidently saw himself as reforming Cavalieri's doctrines to bring them within the purview of what he termed his "method of motions."

It is well known that Leibniz was profoundly influenced by his reading of Hobbes, and he seems to have been particularly enamored of the Hobbesian concept of conatus. In his famous 1670 letter to Hobbes, Leibniz declares the English philosopher to be "wholly justified" in "the foundations [he has] laid concerning the abstract principles of motion" (Leibniz to Hobbes, 22 July, 1670; GP, 7: 573). To the extent that the concept of *conatus* is the basis for Hobbes's analysis of motion, this endorsement suggests that Leibniz was ready to follow Hobbes in using the concept for the analysis of all phenomena produced by motion. Indeed, scholars today are generally accept that "Leibniz's early writings on natural philosophy are virtually steeped in De Corpore" (Bernstein 1980, 29). In particular, Leibniz's reading of Hobbes appears to have been the source for much of his (admittedly limited) mathematical knowledge before his stay in Paris in the 1670s (Hoffman 1972, 6-8).

The clearest evidence of Hobbes's influence on Leibniz is in his essay *Theoria motus abstracti*, where Leibniz employs the concept of conatus to investigate the nature of motion and eventually arrives at the remarkable conclusion that every body is a momentary mind. In a 1671 letter to Henry Oldenburg, Leibniz announced that his theory of abstract motion provides the basis for the solution of any number of mathematical and philosophical puzzles. The theory, he claimed, "explains the hitherto unresolved difficulties of continuous composition, confirms the geometry of indivisibles and arithmetic of infinities; it shows that there is nothing in the realm of nature without parts; that the parts of any continuum are in fact infinite; that the theory of angles is that of the quantities of unextended bodies; that motion is stronger than motion, and conatus stronger than conatus -- however, conatus is instantaneous motion through a point, and so a point may be greater than a point" (Oldenburg 1965-77, 8: 22).

The "geometry of indivisibles" and the "arithmetic of infinities" to which Leibniz refers are, I take it, the works of Cavalieri and John Wallis. Cavalieri's method of indivisibles is mentioned explicitly in section six of the Theoria motus abstracti, as a theory whose "truth is obviously demonstrated so that we must think of certain rudiments, so to speak, or beginnings of lines and figures, as smaller than any given magnitude whatever." (GP 4: 228). Wallis's 1655 treatise Arithmetica Infinitorum, although not mentioned explicitly in the text, is evidently referred to in the letter to Oldenburg when Leibniz refers to the "arithmetic of infinities". In light of this, it is no great interpretive leap to see Leibniz connecting the doctrine of conatus with the classic problem of quadrature, just as Hobbes had done, and thus to find part of the origin of the calculus in Leibniz's close reading of *De Corpore*.

It would doubtless be going to far to claim that the whole of Leibniz's calculus is simply the application of Hobbes's ideas. It is well known that Leibniz's mathematical thought was also strongly influenced by Galileo's approach to the geometry of indivisibles, for example, and the influence of Huygens cannot be overlooked. Nor, for that matter, can Pascal's investigations into infinite sums and differences. All of these are, without question, part of the background to Leibniz's calculus. Nevertheless, we can agree that Hobbes was one among many whose writings stimulated the development of the Leibnizian approach to the calculus. However, there is one important difference between the Leibnizian and Hobbesian conceptions of conatus that is significant: Leibniz's language (at least in the *Theoria motus abstracti*) strongly implies that *conatus* be a literally infinitesimal quantity, while Hobbes regards it as having finite magnitude, but one so small as to be disregarded. In the end, however, Leibniz adopted a doctrine not far removed from Hobbes's.

2. Incomparable Magnitudes and the Question of Rigor

Traditional criteria of rigorous demonstration forbid the use of infinitary methods, and the standard formulation of Leibniz's calculus certainly seems to run afoul of such restrictions. Mysterious terms *dx* and *dy* appear in equations for curves and increments, only to vanish when their work is done, seeming to hover between something and nothing. It is therefore no great surprise that "traditionalist" opponents would make a case against the *calculus differentialis*, charging Leibniz and his associates with violating standards of rigor that guarantee the security and demonstrative status of mathematics. In replying to these critics, Leibniz employed something very much like Hobbes's notion of points and *conatus* as finite but negligible quantities, although he phrased his defense in terms of "incomparably small" magnitudes.

In reply to the criticisms voiced by Bernard Nieuwentijt, who had held that the infinitesimal quantities *dx* and *dy* were illegitimately dismissed from calculations, Leibniz declared such quantities "incomparably small" and legitimate objects of mathematical study. To Nieuwentijt's requirement that only those quantities are equal whose difference is zero, Leibniz insisted

> I think that those things are equal not only whose difference is absolutely nothing, but also whose difference is incomparably small; and although this difference need not be called absolutely nothing, neither is it a

> quantity comparable with those whose difference it is. Just as when you add a point of one line to another line or a line to a surface you do not increase the magnitude, it is the same thing if you add to a line a certain line, but one incomparably smaller. Nor can any increase be shown by any such construction. (*GM*, 5: 322)

There is an obvious parallel between such "incomparably small" elements of lines or surfaces Hobbes's conception of points, for it is exactly the hallmark of Hobbes's points that -- though finite -- they are too small to be considered in any demonstration. Leibniz's preference here for the language of the incomparable rather than the infinitesimal raises the question of whether such incomparable magnitudes are to be thought of as literally infinitesimal or whether they should be treated as finite but negligible quantities in the manner of Hobbes's points.

At first sight, one might take Leibniz's reply to Nieuwentijt as defending the reality of infinitesimals, seeing the term "incomparably small" as a kind of euphemism for "infinitesimal." But I think such an interpretation ultimately fails. Leibniz declares that it is enough to show that incomparably small quantities can be justly neglected in a calculation, and he refers to certain "lemmas communicated by me in February of 1689" for the full justification of this procedure (*GM* 5: 322). These lemmas of 1689 are contained in Leibniz's *Tentamen de motuum coelestium causis* (*GM*, 6: 144- 160). But when we turn to them for enlightenment, two points become tolerably clear. First, these "incomparable" quantities were intended explicitly to avoid references to infinitesimals and instead to replace infinitesimal magnitudes with finite differences sufficiently small to be ignored in practice. Second, the doctrine of the incomparable has a very strong analogy with Hobbes's treatment of points, *conatus*, and *impetus*. The paragraph expounding these lemmas opens with the declaration that

> I have assumed in the demonstrations *incomparably small* quantities, for example the difference between two common quantities which is incomparable with the quantities themselves. Such matters as these, if I am not mistaken, can be set forth most lucidly in what follows. And then if someone does not want to employ *infinitely small* quantities, he can take them to be as small as he judges sufficient to be incomparable, so that they produce an error of no importance and even smaller than any given [error]. Just as the Earth is taken for a point, or the diameter of the Earth for a line infinitely small with respect to the heavens, so it can be demonstrated that if the sides of an angle have a base incomparably less than them, the comprehended angle will be incomparably less than a rectilinear angle, and the difference between the sides will be incomparable with the sides themselves; also, the difference between the whole sine, the sine of the complement, and the secant will be incomparable to these differences. (*GM*, 6: 150-1)

The use intended for such incomparably small magnitudes is to avoid disputes about the nature or existence of infinitesimal quantities, and Leibniz holds that "it is possible to use ordinary [communia] triangles similar to the unassignable ones, which have a great use in finding tangents, maxima, minima, and for investigating the curvature of lines." (*GM*, 6: 150) In other words, the lemmas on incomparable magnitudes are to serve as a foundation for the calculus which permits the talk of infinitesimals to be reinterpreted in terms of incomparable (but apparently finite) differences. These lemmas loom large in Leibniz's writings on the foundations of the calculus, since he almost invariably refers back to them in later discussions on the nature of the infinitesimal. It is also significant that the incomparably small satisfies Hobbes's definition of a geometric point -- it is a quantity sufficiently small that its magnitude cannot be regarded in a demonstration.

3. Fictional Infinitesimals and Incomparable Magnitudes.

When we turn to Leibniz's treatment of the foundations of the calculus after 1700, the theme of the fictionality of the infinitesimal becomes more clearly defined. There were two controversies in the Parisian Academie des Sciences that drew Leibniz into a discussion of the nature of infinitesimals, and in both cases he elaborated a theory in which the infinitesimal turns out to be a fictional entity, albeit a fiction that is sufficiently well-grounded that it cannot lead from true premises to a false conclusion. The first of these controversies was initiated by Michel Rolle, who argued that the notion of an infinitesimal was not only inconsistent, but that the calculus that employed it could lead to error. The second controversy concerned the logarithms of negative numbers and pitted Leibniz against Jean Bernoulli. I lack the time to go into either of these in detail, but Leibniz's pronouncements offer a chance to see the ultimate status of his theory of the infinitesimal.

In a famous letter to M. Pinson, parts of which were published in the *Journal de Sçavans* in 1701, Leibniz offered his opinion on the controversy initiated by Rolle. In the letter, he responded to an anonymous criticism of the infinitesimal which Abbé Gouye had published in the *Journal*. Leibniz argued in reply that

> there is no need to take the infinite here rigorously, but only as when we say in optics that the rays of the sun come from a point infinitely distant, and thus are regarded as parallel. And when there are more degrees of infinity, or infinitely small, it is as the sphere of the earth is regarded as a point in respect to the distance of the sphere of the fixed stars, and a ball which we hold in the hand is also a point in comparison with the semidiameter of the

> sphere of the earth. And then the distance to the fixed stars is infinitely infinite or an infinity of infinities in relation to the diameter of the ball. For in place of the infinite or the infinitely small we can take quantities as great or as small as is necessary in order that the error will be less than any given error. In this way we only differ from the style of Archimedes in the expressions, which are more direct in our method and better adapted to the art of discovery. (*GM*, 4: 95-96)

These remarks are of a piece with Leibniz's earlier claims about the eliminability of infinitesimal magnitudes: he denies that the calculus really needs to rely upon considerations of the infinite and again insists that it can be based on a procedure of taking finite but "negligible" errors that can be made as small as desired. And again, it is worth observing that Hobbes used essentially the same language, comparing the earth to a point in comparison to the heavens.

The more ardent partisans of the infinitesimal (notably Jean Bernoulli, Varignon, and the Marquis de L'Hôpital) were deeply concerned by Leibniz's apparent concession to the critics of the calculus. Varignon wrote to Leibniz in November of 1701 requesting a clarification of Leibniz's views on the reality of infinitesimals and expressing the fear that the publication of the letter to M. Pinson had done harm to the cause because some had taken him to mean that the calculus was inexact and capable only of providing approximations. Varignon therefore requested "that you send us as soon as possible a clear and precise declaration of your thoughts on this matter." (Varignon to Leibniz, 28 November, 1701, *GM*, 4: 90).

In his reply to Varignon Leibniz issued a summary statement of his views on the infinite and its role in the calculus. This statement brings together themes we have already seen: the fictional nature of infinitesimals, the possibility of basing the calculus upon a science of incomparably small (but still finite) differences, and the equivalence of the new methods and the Archimedean techniques of exhaustion. After assuring Varignon that his intention was "to point out that it is unnecessary to make mathematical analysis depend on metaphysical controversies or to make sure that there are lines in nature which are infinitely small in a rigorous sense" (Leibniz to Varignon, 2 February, 1702; *GM* 4: 91), Leibniz once again suggests that incomparably small magnitudes be taken in place of the genuine infinite. These incomparables would provide "as many degrees of incomparability as one may wish;" and although they are really finite quantities they can still be neglected, in accordance with the notorious "lemmas on incomparables" from the Leipzig *Acta*. (Leibniz to Varignon 2 February, 1702; *GM*, 4: 91-2). Leibniz's account of the nature of infinitesimals thus brings us again to the lemmas on incomparable magnitudes in the *Tentamen de motuum coelestium causis*. But, as I have noted, this account of the incomparably small seems very much of a piece with Hobbes's notion of *conatus*. It is characteristic of the fictional infinitesimal that it is a "well founded" fiction, by which Leibniz means that indulgence in the fiction will not produce error. As Leibniz explained to Varignon, both infinitesimal magnitudes and imaginary roots have a foundation in the nature of things, and the world is structured as if there were such things, though in reality there are none. Indulging in the fiction is therefore harmless, and even useful, since it encourages economy of expression and can stimulate research into new results.

The full scope of this "fictionalist" reading of the infinite was not made widely known, largely because Leibniz and his associates had reason to fear that any public retreat from a full commitment to the reality of the infinitesimal would complicate the already difficult battle for the acceptance of the calculus. As Leibniz explained in a late letter "When our friends were disputing in France with the Abbé Gallois, father Gouye and others, I told them that I did not believe at all that there were actually infinite or actually infinitesimal quantities; the latter, like the imaginary roots of algebra $\sqrt{-1}$ were only fictions, which however could be used for the sake of brevity or in order to speak universally. . . But as the Marquis de l'Hôpital thought that by this I should betray the cause, they asked me to say nothing about it, except what I already had said in the Leipzig *Acta*. (Leibniz to Dangicourt, 11 September 1716; Dutens, 3: 500-501)

In correspondence with Des Bosses, Leibniz added that "Philosophically speaking, I no more admit magnitudes infinitely small than infinitely great . . . . I take both for mental fictions, as more convenient ways of speaking, and adapted to calculation, just like imaginary roots are in algebra. I once demonstrated that these expressions have a great use both in abbreviating thought and aiding discovery, and that they cannot lead to error, since in place of the infinitely small one may substitute [a quantity] as small as one wishes, and since any error will always be less than this, it follows that no error can be given. But the Reverend Father Gouyé, who objected, seems not to have understood me adequately." (Leibniz to Des Bosses, 11 March, 1706; GP, 2: 305)

The final piece in the puzzle of Leibniz's theory of the infinitesimal, and one that leads us back to Hobbes, is the late note *Observatio quod rationes sive proportiones non habeant locum circa quantitates nihilo minores, et de vero senso methodi infinitesimalis*, which appeared in the *Acta Eruditorum* in April of 1712. It was sparked by a controversy over the nature of ratios between positive and negative quantities, which grew to include the cases of logarithms and roots of negative numbers. Jean Bernoulli (who was also a firm believer in the reality of infinitesimals and a chief partisan in favor of the Leibnizian calculus in the Academie) held that logarithms of negative numbers were the same as those of positive numbers, so that the logarithm of $-a$ is the same a the logarithm of a. Leibniz treated the issue of negative quantities in ratios, logarithms, and roots as fictions that could be harmlessly employed in calculation, but which did not correspond to anything mathematically real. In Leibniz's view, there is no ratio of +1 to − 1 (as Bernoulli required), since otherwise it would be the same as the ratio of −1 to +1.

Likewise, the fictionality of the infinitesimal is stated in language that seems to have been almost borrowed from Hobbes.

In objecting to the notion that there could be a proper ratio between positive and negative quantities, Leibniz remarked:

> just as I have denied of the reality of a ratio, one of whose terms is less than zero, I equally deny that there is properly speaking an infinite number, or an infinitely small number, or any infinite line or a line infinitely small.... The infinite, whether continuous or discrete, is not properly a unity, nor a whole, nor a quantity, and when by analogy we use it in this sense, it is a certain *facon de parler*; I should say that when a multiplicity of objects exceeds any number, we nevertheless attribute to them by analogy a number, and we call it infinite. And thus I once established that when we call an error infinitely small, we wish only to say an error less than any given, and thus nothing in reality. And when we compare an ordinary term, an infinite term, and one infinitely infinite, it is exactly as if we were to compare, in increasing order, the diameter of a grain of dust, the diameter of the earth, and that of the sphere of the fixed stars....(GM V, 389)

One striking feature of this late publication is Leibniz's reminiscence about his Paris period. Leibniz recalls his encounters with the work of Arnauld, Wallis, and Joachim Jung in the 1670s, and it is precisely during this period that Leibniz was working on the *Theoria motus abstracti* and still very much under the influence of Hobbes. As Marc Parmentier has put it, "the first lines of his article incline one to think that the recent polemic [over the nature of ratios] had revived a personal recollection that forty years of intense diplomatic, scientific, and historical activity had not been able to erase, and which suddenly came into his memory with its original clarity" (423).

4. Conclusions.

This very brief account of Leibniz's doctrine of the ficitionality of the infinitesimal raises perhaps more questions than it answers. I would like to close by considering two important consequences of Leibniz's doctrine of the fictional infinitesimal. The first is the question of how Leibniz might guarantee that the infinitesimal is, indeed, a well-founded fiction. The second, and related, issue is whether there is a stable conception of mathematical that underlies Leibniz's writings on the calculus.

A fiction is well-founded in the Leibnizian sense when it does not lead us astray, so that indulgence in the fiction is harmless. The basic idea here seems to be something to the effect that we can "speak with the vulgar" when we employ the language of the infinitesimal, but "think with the learned" when we recognize that there really are no such things. Yet we still stand in need of some sort of guarantee that we will not, in fact, be led astray. In the mathematical context, this means that we need some kind of proof to the effect that infinitesimals can always, at least in principle, be eliminated and reasoning that depends on them can be replaced by reasoning that considers only finite differences between finite quantities. Leibniz often makes grand programmatic statements to the effect that derivations which presuppose infinitesimals can always be re-cast as exhaustion proofs in the style of Archimedes. But Leibniz never, so far as I know, attempted anything like a general proof of the eliminability of the infinitesimal, or offered anything approaching a universal scheme for re-writing the procedures of the calculus in terms of exhaustion proofs. The closest thing we have are the notorious "lemmas on incomparable magnitudes" from 1689, but these are really more promissory notes with a serious admixture of hand-waving rather than rigorous proofs. What, then, are we to make of Leibniz's confidence that the infinitesimal is a well-founded fiction? He was certainly aware that some infinitesimal arguments could lead to paradox and contradiction, but it is unclear whether he had a surefire way of avoiding error.

A related issue is what the Leibnizian conception of mathematical rigor really looks like. As classically understood, a rigorous argument is one that begins with transparently true first principles, proceeds by valid inference procedures, and deals only with objects that are clearly conceived. It is far from clear whether Leibniz would allow that the proof procedures of the calculus are, in fact, rigorous in this sense. After all, the infinitesimal is not the sort of thing we can conceive clearly, and it seems a bit odd to think that there might be transparently true first principles that deal with merely fictional objects. In the end, then, we might ask whether classifying infinitesimals as "useful fictions" can really deflect the criticism of the calculus which characterizes it as unrigorous. This is not an issue I'm in any position to resolve at the moment, and will leave it for another day